T_m), whereas the method using Eq. (7) has an error of $\Delta A/A + \Delta\epsilon/\epsilon + 6(\Delta T_m/T_m)$.

Concluding Remarks

The measuring technique described in this paper has several desirable characteristics and appears capable of yielding absorptance and total hemispherical emittance over a wide range of temperatures. The desirable features exhibited by the proposed experimental method are: 1) α, ϵ, and α/ϵ can be determined independently and simultaneously from one set of data. 2) By the proper selection of radiation intensity, a range of sample temperatures can be provided. 3) The measurement accuracy is promising.

A sample calculation for a polished copper sample, 1-mil thick for reasonable temperature and phase measurement precision, shows the error to be less than 7% for emittance and 10% for absorptance when a cyclic variation in radiation less than 4 cycles/hr is used. In general, the technique error is temperature and material dependent so that the preciseness of the method must finally be evaluated on particular requirements.

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An Analysis of the "Laminar Instability" Problem in Gas-Cooled Nuclear Reactor Passages

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Nuclear reactor passages in laminar flow have long been regarded as unstable in that region of operation where the rate of change of pressure drop with weight flow at constant heat input is negative. In the present paper a time-dependent stability analysis of the problem affirms that this criterion is valid for excursions at constant pressure drop and shows that the instability is associated with the thermal response of the reactor core to an infinitesimal disturbance. The analysis also allows the calculation of a complete excursion at constant pressure drop from an initially unstable laminar equilibrium point. An excursion toward higher flow rate will continue until the high flow rate for the same pressure drop and heat rate to the gas is realized, probably under conditions of turbulent flow. An excursion to lower flow rate will progress until there is tube burnout. With increase in the heat capacity of the reactor core, the time for a complete excursion increases. The turbulent flow equilibrium points are always stable to excursions at constant pressure drop. Stability criteria are also presented for excursions at other than constant pressure drop. Throughout, the method of analysis is such that growth rates may be obtained for any reactor whose steady-state conditions are known whether they be for uniform heating, cosine heating, or any form of zoned heating.

Nomenclature

A = passage flow area
$\mathcal{A}$ = cross-sectional area of reactor core associated with single passage
c = specific heat of core material
C = constant in friction factor, Eq. (A14)
c_p = specific heat at constant pressure of coolant
f = friction factor
h = heat-transfer coefficient
L = length of reactor passage
m = viscosity-temperature exponent ($\mu \sim T^m$)
$\mathfrak{m}$ = density of reactor core material
n = exponent of dependence of friction factor on Reynolds number, Eq. (A14)
p = pressure
Q = heat-transfer rate to coolant gas
$q(x)$ = normalized distribution of heat transfer to gas
R = gas constant
r_h = hydraulic radius
$\mathfrak{T}$ = dimensionless time unit, Eq. (26)
T = gas temperature
T_w = reactor core temperature
t = time
w/A = flow rate per unit area
x = longitudinal distance along flow passage
α = denominator in definition of growth rate
β = growth rate
γ = numerator in definition of growth rate
μ = viscosity

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ϕ = reactor energy release rate
τ = final to initial gas-temperature ratio

Subscripts

0 = value at equilibrium point
1 = conditions at entrance to flow passage
L = conditions at exit of flow passage
avg = average

Introduction

IT is a property of the steady laminar flow with heat addition in constant-area passages that for a fluid whose viscosity increases with temperature there are two mass flow rates possible for a given pressure drop and heat input rate. Such passages have long been regarded as unstable in that region of operation where the rate of change of pressure drop with weight flow at constant heat input is negative.[1-3] However, the existing literature does not detail any time-dependent stability analysis in support of this criterion.† Nor is any clear description given of the dynamics of the instability and the associated mechanisms of instability. Bussard and Delauer[3] attribute the instability to the "negative" resistance implied by the foregoing criterion. Gruber and Hyman[2] argue that the flow is unstable in the sense that an increase in the heat input tends to stop the flow. Harry,[5] in a more recent consideration of the steady-state picture, suggests that "the instability is not related to high-frequency gasdynamics and thus by inference is related to the wall capacity and heat input."

Prompted by the vagueness and lack of unanimity in the description of the phenomenon, it is the objective of the present paper to ascertain through a time-dependent analysis whether there is a stability problem here and if so to determine its nature in the sense of identifying the factors that influence the condition of neutral stability and the growth rates. It may be remarked at this point that the negative slope criterion of Ledinegg[1] is in fact verified for excursions at constant pressure drop, but the present analysis further enables the calculation of the complete history of flow excursions from the unstable equilibrium point.

The analysis is formulated in such a way that it may be applied to any reactor whose steady-state conditions are known whether they be for laminar or turbulent flow, and whether they be for uniform heating, cosine heating, or any form of zoned heating. The time-dependent analysis will be preceded by a discussion of the nature of the steady-state conditions in gas-cooled reactor passages.

Steady-State Solutions

The steady one-dimensional flow of a perfect gas in a constant-area flow passage with friction and heat addition was considered by Harry (Appendix B of Ref. 5) for the very low Mach number flows of interest in this paper. Harry's result for the case of uniform heat addition is

$$p_{\text{avg}}\Delta p = RT_1\left(\frac{w}{A}\right)^2\left[(\tau - 1) + \frac{f_0\mu_1{}^nL}{[4r_h(w/A)]^n2r_h(mn + 2)}\,\frac{\tau^{mn+2} - 1}{\tau - 1}\right] \quad (1)$$

where Δp is the pressure drop, p_{avg} is the arithmetic average of the inlet and exit pressure, τ is the exit-to-entrance gas-temperature ratio, m is the exponent of the viscosity-temperature relation $\mu \sim T^m$, and n is the exponent in the relation between friction factor and Reynolds number. For laminar flow $n \approx 1$ whereas for turbulent flow $n \approx 0.2$.

To determine the variation of pressure drop with flow rate it is instructive to look at the form of Eq. (1), where the temperature ratio τ is written in terms of the rate of heat addition $\bar{Q}$ and the flow rate per unit area (w/A). For $\tau \gg 1$ this yields

$$p_{\text{avg}}\,\Delta p = RT_1\left(\frac{\bar{Q}L}{r_hc_pT_1}\right)\left[\left(\frac{w}{A}\right) + \frac{f_0\mu_1{}^nL}{(4r_h)^n2r_h(mn + 2)}\,\frac{(\bar{Q}L/r_hcpT_1)^{mn}}{(w/A)^{mn+n-1}}\right] \quad (2)$$

The first term in the bracketed expression represents the momentum pressure drop (due just to heat addition) whereas the second is the frictional pressure drop. For laminar flow of a gas ($n = 1$, $m \approx 0.6$–0.7), the quantity $(mn + n - 1)$ is positive. Thus for large flow rates the frictional drop becomes insignificant compared to the momentum pressure drop. However, for low flow rates the frictional pressure drop dominates. As the flow rate is reduced toward zero, the momentum pressure drop becomes insignificant while the frictional pressure drop increases as $(w/A)^{-m}$. For the flow of a single species of constant exponent m, the pressure drop would tend to become infinite in the limit of zero flow rate. These trends have been cited by Bussard and Delauer[3] and also apply to nonuniform heating. For turbulent flow ($n = 0.2$) the pressure drop increases monotonically with flow rate. Thus the steady-state characteristics of coolant passages are as shown in Fig. 1 taken from Harry's report.[5] For a given entrance pressure and temperature, the inlet Mach number M_1 is a direct index of the flow rate since

$$(w/A) = M_1p_1(\gamma/RT_1)^{1/2} \quad (3)$$

The recent experiments of Turney, Smith, and Juhasz[6] and of Kolbe[7] confirm the predicted existence of the two flow rates for a given pressure drop and heat-transfer rate to the gas.‡

In the case of a diatomic coolant gas, say that for a given heat-transfer rate to the passage, the flow rate is reduced to such an extent that the coolant gas dissociates. This may lead to some irregularity in the curve of Δp vs (w/A). However, once dissociation is substantially complete, namely that it occurs near the tube entrance, the behavior again becomes that of Eq. (2) for the single monatomic gas species.

Time-Dependent Analysis

Assumptions

The analysis is carried out under the following premises:

1) The flow in the reactor passages is adequately described by one-dimensional flow relations that include heat addition and friction.

2) Hydrodynamic instabilities of the nature associated with nonuniform velocity and temperature profiles are unimportant. In fact, the one-dimensional treatment precludes the consideration of hydrodynamic instabilities.

3) Since the Mach numbers in reactor passages are generally very much less than unity (they are more likely to be of order 10^{-3} to 10^{-1}), the flow time or residence time is very long compared to the time required for acoustic disturbances to travel the length of the heated reactor tubes. The residence time is in turn small compared to the thermal response time of the portion of the reactor core associated with a

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† Bussard and Delauer[3] allude that the criterion has in fact been verified in unpublished work by Longmire at the Los Alamos Scientific Laboratory. Longmire's analysis (as described by Bankston[4]) differs from the present one in that he ignores heat exchange with the core, whereas in the present study heat exchange with the core is the dominant effect.

‡ The data of Guevara, McInteer, and Potter[8] would probably have shown more extensive evidence of the two flow rates had the authors presented their results in terms of heat-transfer rate to the gas rather than electrical power input to their resistively heated tube.

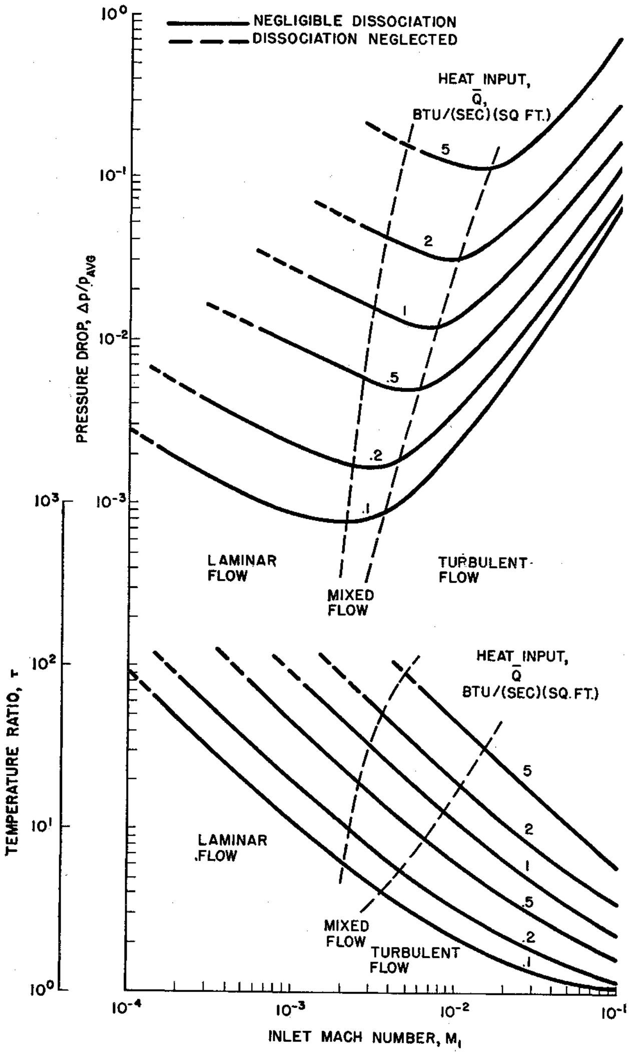

Fig. 1 Steady-state passage characteristics. Hydrogen-inlet pressure, 20 psia; inlet temperature, 50°R; tube diameter, 0.10 in.; tube length 4 ft.

given flow passage. If it is now assumed that thermal coupling between the coolant flow and the reactor core is important, then the thermal response time of the core becomes the characteristic time of the problem. The time-dependent terms in the flow equations become negligibly small and only the time-dependent term in the equation governing the energy balance of the core is retained. The one-dimensional transient flow in the tubes may thus be regarded as a continuous succession of steady-state flows governed only by those terms appearing in the steady-state flow equations.§

4) The specific heats of the reactor coolant gas are constant. The specific heat of the reactor core material may however vary with core temperature.

Governing Equations

The quantities of this problem—the flow variables and the fluid properties—are generally dependent on both x and t. Heat-exchanger performance, however, is very often given in terms of the "integral" properties of the heat exchanger—those obtained by integrating over x. The integral properties of a given tube in a heat exchanger are the mass flow; the pressure drop; the enthalpy (or temperature) ratio, final to inital; the average heat rate to the gas; and some characteristic wall temperature. The equations governing such integral properties under the assumptions of this problem can be dependent only on time. The procedure, therefore, is to formulate the equations governing the integral properties and study their response to a small disturbance about an equilibrium point. It will develop that these equations reduce to a single, first-order, ordinary differential equation describing the average heat-transfer rate to the gas $\bar{Q}$ as a function of time.

In accordance with assumption 3 in the foregoing, the continuity, momentum, and energy equations for the flow in the tube contain only the terms of the steady-flow equations. For a constant-area tube the continuity equation is nothing more than a statement that the mass flow per unit area (w/A) is constant over the length of the coolant passage but can be a function of time.

The momentum relation is also a "steady-state" relation, and for any heat-exchanger tube whose steady-state operation is known (Fig. 1 for example), the momentum relation is available in "integral" form simply as

$$\Delta p = \Delta p(w/A,\bar{Q}) \tag{4}$$

That is, the pressure drop is a known function of both the flow rate and average heat rate to the gas. In general, since (w/A) and $\bar{Q}$ may be time-dependent, therefore Δp is also time-dependent. However it is also possible to have an excursion in which (w/A) and Q vary in such ways that Δp is time-independent.

The steady-state energy equation is

$$(w/A)(\partial H/\partial x) = Q/r_h \tag{5}$$

In this equation H is the stagnation enthalpy. Since the Mach number range is well below unity, the stagnation enthalpy is approximately equal to the static enthalpy here idealized as c_pT. The quantity Q is the local heat rate per unit area and r_h is the hydraulic radius. With these approximations the energy equation (5) is rewritten

$$\frac{w}{A}(t)c_p\,\frac{\partial T(x,t)}{\partial x} = \frac{Q(x,t)}{r_h} \tag{5a}$$

This equation is not yet in integral form. Accordingly it is integrated with respect to x

$$\frac{w}{A}(t)c_p\int_0^L \frac{\partial T(x,t)}{\partial x}\,dx = \frac{1}{r_h}\int_0^L Q\,dx$$

yielding

$$(w/A)c_p(T_L - T_1) = \bar{Q}(L/r_h) \tag{6}$$

where $\bar{Q}$ is the average heat rate to the gas and is defined

$$\bar{Q} \equiv \frac{1}{L}\int_0^L Q\,dx \tag{7}$$

The equation governing the energy balance of the core is

$$c(T_w)\,\frac{\partial T_w(x,t)}{\partial t} = \frac{\varphi(x,t)}{\mathfrak{m}} - \frac{Q(x,t)A}{\mathfrak{m}\mathfrak{a}r_h} \tag{8}$$

where $\mathfrak{m}$ is the density of the reactor core, $\mathfrak{a}$ is the portion of the cross section of the core assigned to a given flow passage, $c(T_w)$ is the specific heat of the core material, the core temperature $T_w(x,t)$ is assumed not to vary in a direction perpendicular to the flow direction, and $\varphi(x,t)$ is the nuclear energy release rate per unit volume of the reactor. The

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§ In a private communication, D. P. Harry indicated that he made a numerical study of response to high-frequency disturbances using "method of characteristics" solutions to the complete one-dimensional unsteady-flow equations and found all coolant flow conditions—laminar, turbulent, and mixed—to be stable. He attested also to the validity of assumptions 3 for the study of flow excursions.

axial thermal conduction term is omitted in Eq. (8) as it is assumed to be small compared to the other terms of the equation.

Equation (8) is not yet in integral form. But first, in order to close the set of equations, we require the relationship for heat transfer from the core to the gas, namely

$$Q(x,t) = h(x,t)[T_w(x,t) - T(x,t)] \qquad (9)$$

where $h(x,t)$ is the heat-transfer coefficient.

We proceed by differentiating (9) with respect to time and substituting from Eq. (8)

$$\frac{\partial Q}{\partial t} = \frac{h}{\mathfrak{m}c}\left(\varphi - \frac{QA}{\mathfrak{a}r_h}\right) - h\frac{\partial T}{\partial t} + Q\frac{\partial \ln h}{\partial t} \qquad (10)$$

Integration with respect to x over the length of the passage yields the form

$$\frac{d\bar{Q}}{dt} = \frac{1}{\mathfrak{m}L}\int_0^L \frac{h}{c}\varphi dx - \frac{A}{\mathfrak{m}\mathfrak{a}r_hL}\int_0^L \frac{h}{c}Qdx - \frac{1}{L}\int_0^L h\frac{\partial T}{\partial t}dx + \frac{1}{L}\int_0^L Q\frac{\partial \ln h}{\partial t}dx \qquad (11)$$

The details of evaluating the four integrals on the right side of Eq. (11) are given in Appendix A. It is shown that subject to the stated assumptions plus the condition that the inlet temperature is fixed, Eq. (11) may be written

$$\frac{d\bar{Q}}{dt}\left\{1 + (\tau - 1)\left[\frac{\bar{\bar{h}}T_1}{\bar{Q}}\frac{d\left(\frac{\bar{T}}{T_1}\right)}{d\tau} - \frac{\partial \widetilde{\ln h}}{\partial\tau}\right]\times\right.$$
$$\left.\left[1 + \frac{\bar{Q}}{\left(\frac{w}{A}\right)}\frac{\left(\frac{\partial(\Delta p)}{\partial\bar{Q}}\right)_{w/A}}{\left(\frac{\partial(\Delta p)}{\partial(w/a)}\right)_{\bar{Q}}}\right] + \frac{\partial\widetilde{\ln h}}{\partial\ln\left(\frac{w}{A}\right)}\frac{\bar{Q}}{\left(\frac{w}{A}\right)}\frac{\left(\frac{\partial(\Delta p)}{\partial\bar{Q}}\right)_{w/A}}{\left(\frac{\partial(\Delta p)}{\partial(w/A)}\right)_{\bar{Q}}}\right\} =$$
$$\frac{1}{\mathfrak{m}}\frac{A}{\mathfrak{a}r_h}\left(\frac{\bar{h}}{c}\right)\left[\frac{\bar{\varphi}ar_h}{A} - \bar{Q}\right] + \frac{\bar{Q}}{\left(\frac{w}{A}\right)}\times$$
$$\left\{(\tau - 1)\left[\frac{\bar{\bar{h}}T_1}{\bar{Q}}\frac{d\left(\frac{\bar{T}}{T_1}\right)}{d\tau} - \frac{\partial\widetilde{\ln h}}{d\tau}\right] + \frac{\partial\widetilde{\ln h}}{\partial\ln\left(\frac{w}{A}\right)}\right\}\times\frac{\frac{d(\Delta p)}{dt}}{\left[\frac{\partial(\Delta p)}{\partial\ln(w/A)}\right]_{\bar{Q}}} \qquad (12)$$

where

$$\tau = T_L/T_1$$

and the averaged quantities $\bar{\bar{h}}$, $(\bar{T}/T_1)$, $\widetilde{\ln h}$, and $(\bar{h}/c)$ are as defined in Appendix A. Equation (12) is the equation governing the transient behavior of a reactor passage and the portion of the core associated with it.

Reactor Passage Excursions at Constant Δp

A reactor core in steady-state operation has a certain pressure difference across the tubes. This pressure difference is the same for all tubes connected to common headers. However, since for a given pressure difference and average heat flux $\bar{Q}$ there are two possible flow rates, it is possible that not all tubes are the same operating point. In this section the temporal behavior of the "errant" flow passage is examined under the assumption that the pressure difference between the headers is unaffected by the behavior of a few errant flow passages.

Recognizing that

$$\frac{\bar{Q}}{\left(\frac{w}{A}\right)}\frac{\left[\frac{\partial(\Delta p)}{\partial\bar{Q}}\right]_{w/A}}{\left[\frac{\partial(\Delta p)}{\partial(w/A)}\right]_{\bar{Q}}} = -\left.\frac{d\ln\left(\frac{w}{A}\right)}{d\ln\bar{Q}}\right]_{\Delta p} \qquad (13)$$

the form of Eq. (12) appropriate to an excursion at constant Δp is

$$\frac{d\bar{Q}}{dt}\left\{1 + (\tau - 1)\left[\frac{\bar{\bar{h}}T_1}{\bar{Q}}\frac{d\left(\frac{\bar{T}}{T_1}\right)}{d\tau} - \frac{\partial\widetilde{\ln h}}{\partial\tau}\right]\times\right.$$
$$\left.\left[1 - \frac{d\ln\left(\frac{w}{A}\right)}{d\ln\bar{Q}}\Bigg)_{\Delta p}\right] - \frac{\partial\widetilde{\ln h}}{\partial\ln\left(\frac{w}{A}\right)}\frac{d\ln\left(\frac{w}{A}\right)}{d\ln\bar{Q}}\Bigg)_{\Delta p}\right\} =$$
$$\frac{1}{\mathfrak{m}}\frac{A}{\mathfrak{a}r_h}\left(\frac{\bar{h}}{c}\right)\left[\frac{\bar{\varphi}\mathfrak{a}r_h}{A} - \bar{Q}\right] \qquad (14)$$

Stability Criterion—Evaluation of Growth Rates

Consider now the response of a reactor passage at constant Δp and with constant nuclear heat release $\bar{\varphi}$ to a small disturbance about its equilibrium point. Equation (14) may be rewritten as

$$\alpha(t)\frac{d\bar{Q}}{dt} = \gamma(t)\left[\frac{\bar{\varphi}\mathfrak{a}r_h}{A} - \bar{Q}\right] \qquad (15)$$

where $\alpha(t)$ is the bracketed quantity on the left side of (14) and $\gamma(t) = (A/\mathfrak{m}\mathfrak{a}r_h)(\bar{h}/c)$.

Let each quantity in (15) be represented by its equilibrium value (subscript 0) plus a small perturbation (subscript 1), except for $\bar{\varphi}$, which is constant. Then

$$\left.\begin{matrix}\bar{Q} = \bar{Q}_0 + \bar{Q}_1 & \alpha = \alpha_0 + \alpha_1 \\ \gamma = \gamma_0 + \gamma_1 & \bar{\varphi} = \bar{\varphi}_0\end{matrix}\right\} \quad \text{where } (\ )_1 \ll (\ )_0 \qquad (16)$$

Substitution of (16) into (15) yields

Zero order:

$$\bar{\varphi}_0\mathfrak{a}r_h/A = \bar{Q}_0 \qquad (17)$$

First order:

$$\alpha_0(d\bar{Q}_1/dt) = -\gamma_0\bar{Q}_1 \qquad (18)$$

The zero-order solution states that the equilibrium condition is that where all of the reactor heat release goes into heating the gas. The first-order equation yields the stability criterion. It is readily seen that if one lets $\bar{Q}_1$ vary as $e^{\beta t}$, then the growth rate β is given by

$$\beta = -(\gamma_0/\alpha_0) \qquad (19)$$

A positive value of β indicates growth or instability whereas a negative value of β indicates that a small disturbance will decay and the equilibrium condition (17) is restored. In either case it is seen from Eq. (18) that the growth or decay is nonoscillatory. Since γ_0 is always positive, instability can be obtained only when the value of α_0 at the pertinent equilibrium condition is negative.

Examination of the expression for α_0,

$$\alpha_0 = \left\{1 + (\tau - 1)\left[\frac{\bar{\bar{h}}T_1}{\bar{Q}}\,\frac{d\left(\frac{\bar{T}}{T_1}\right)}{d\tau} - \frac{\partial\,\widetilde{\ln h}}{\partial\tau}\right] \times \right.$$

$$\left.\left[1 - \left.\frac{d\ln\left(\frac{w}{A}\right)}{d\ln\bar{Q}}\right)_{\Delta p}\right] - \frac{\partial\,\widetilde{\ln h}}{\partial\ln\left(\frac{w}{A}\right)}\left.\frac{d\ln\left(\frac{w}{A}\right)}{d\ln\bar{Q}}\right)_{\Delta p}\right\} \tag{20}$$

shows that for $(\bar{\bar{h}}T_1/\bar{Q})[d(\bar{T}/T_1)/d\tau] > \partial\,\widetilde{\ln h}/\partial\tau$, a condition that is just about always true, instability occurs whenever

$$\left.\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\Delta p} > 0 \tag{21}$$

Since $[\partial(\Delta p)/\partial\bar{Q}]_{w/A}$ is always positive, then from Eq. (13) instability is obtained for

$$\left[\frac{\partial(\Delta p)}{\partial(w/A)}\right]_{\bar{Q}} < 0 \tag{22}$$

a condition identified with the low-flow-rate, laminar, steady-state solution.

The neutral point is where

$$\left.\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\Delta p} \to \infty \tag{23a}$$

or

$$\left[\frac{\partial(\Delta p)}{\partial(w/A)}\right]_{\bar{Q}} = 0 \tag{23b}$$

For flow rates greater than corresponding to neutral stability, the passage is stable to flow excursions. Also, since for turbulent flow $[\partial(\Delta p)/\partial(w/A)]_{\bar{Q}}$ is always positive, a passage in turbulent flow is always stable.

Equations (21–23) are a confirmation by analysis of the generally assumed criterion for instability at constant pressure drop. Physically, the instability is not in any way hydrodynamic. Rather it is a consequence of the compatibility between the heat-transfer characteristics of the flow passage and the thermal characteristic of the core. This is shown in Fig. 2. The left equilibrium point is unstable. An excursion to higher flow rate requires additional heat to the gas. This is provided through the associated cooling of the core, so the excursion proceeds. In an excursion to lower flow rate the gas absorbs less of the reactor energy release; the core, however, is being heated, accounting for the balance of the reactor energy release and the excursion continues. For the right equilibrium point, an excursion to higher (w/A) would, for compatibility, correspond to a decrease in heat to the gas. However, since the core is being cooled, there is an increase in heat to the gas and the flow is driven back to the equilibrium point. If there is an excursion to lower flow rate, the reactor must supply additional heat both to the gas and to the core. At constant reactor energy release rate ϕ, it cannot do this and so the flow reverts to the equilibrium point. Thus the high-flow-rate equilibrium point is stable.

The growth rate β for any equilibrium point is obtained simply by evaluating Eqs. (19) and (20) for the equilibrium points in question. For the same operating point the magnitude of the growth rate is inverse to the heat capacity of the reactor core.

Calculation of a Complete Flow Excursion

The complete flow excursion for constant Δp is governed by Eq. (14). An excursion is initiated by assuming a small initial displacement of $\bar{Q}$ from its equilibrium value $\bar{Q}_0$. Starting at an unstable equilibrium point, if $\bar{Q}$ is increased slightly above $\bar{Q}_0$ corresponding to an increase in flow rate, the excursion proceeds in the direction of continued increase in flow rate until the high-flow-rate, stable point is reached. Since $\bar{Q}$ first increases and then decreases during an excursion, the integration is carried out more conveniently in terms of the temperature ratio τ, which decreases monotonically during the excursion. If $\bar{Q}$ is decreased slightly from $\bar{Q}_0$ corresponding to a decrease in the flow rate, the excursion proceeds in the direction of continued decrease in flow rate. However, since there is no equilibrium point to approach, the excursion proceeds to tube burnout.

Examples

To illustrate and elaborate on some of the prior statements, examples will be presented for the case of uniformly heated passages. The transient equation and stability criterion for uniformly heated passages are developed in Appendix B. These examples will all be based on the steady-state solutions of Fig. 1.

The first example is that of calculating the growth rates for infinitesimal disturbances for $Q = 0.2$ Btu/ft²-sec. From Eq. (B16)

$$\beta\left\{\frac{m\mathfrak{a}r_h}{A\left(\frac{h_1}{c}\right)_0}\right\} = \frac{-\left[\frac{\tau^{mn+1} - 1}{(\tau - 1)(mn + 1)}\right]_0}{\left\{1 + \left[\frac{h_1T_1}{\bar{Q}}\left(\frac{(mn + 2)(\tau - 1)\tau^{mn+1} - (\tau^{mn+2} - 1)}{(mn + 1)(mn + 2)(\tau - 1)}\right) - mn\left(1 - \frac{\ln\tau}{\tau - 1}\right)\right] \times \left[1 - \left.\frac{d\ln\left(\frac{w}{A}\right)}{d\ln\bar{Q}}\right)_{\Delta p}\right] - (1 - n)\left.\frac{d\ln\left(\frac{w}{A}\right)}{d\ln\bar{Q}}\right)_{\Delta p}\right\}_0} \tag{24}$$

The left-hand side is a dimensionless growth rate. The dimensional growth rate β is inverse to the heat capacity of the reactor core. The results of the calculation are shown in Fig. 3. The neutral point occurs where $[\partial(\Delta p)/\partial(w/A)]_{\bar{Q}} = 0$ at an inlet Mach number of 3×10^{-3}. The high flow rates are damped. The positive growth rates, however, are not monotonic with flow rate. In fact, the maximum growth rate occurs at an inlet Mach number of 2×10^{-3}, with the lower flow rates being even less unstable. For a graphite core with $\mathfrak{a}/A = 5$ operating at the conditions of Fig. 3, the e-folding time at maximum growth rate is about 500 sec ($8\frac{1}{3}$ min). For any other unstable operating point at this value of $\bar{Q}$ the e-folding time is even longer. In general it may be said that the e-folding time is proportional to the heat capacity of the reactor core and inverse to the level of $\bar{Q}$.

In the second and third examples, excursions are calculated about the unstable operating point $\tau = 80$, $\Delta p/p = 5 \times 10^{-3}$, $\bar{Q}_0 = 0.2$ Btu/ft²-sec, $M_1 = 2.25 \times 10^{-4}$. These excursions are at constant Δp. Their trajectories through the steady-state solutions are shown by the dotted lines in Fig. 4.

Consider first the excursion to higher flow rate. The calculation is begun with an initial perturbation of 10% in inlet Mach number corresponding to $\tau_i = 76$. The integral to be evaluated is from eq. (B-15)

$$\mathfrak{J} = \int_{\tau_i}^{\tau} \frac{\frac{d\bar{Q}}{d\tau}\left\{1 + \left[\frac{h_1 T_1}{\bar{Q}}\left(\frac{(mn+2)(\tau-1)\tau^{mn+1} - (\tau^{mn+2}-1)}{(mn+1)(mn+2)(\tau-1)}\right) - mn\left(1 - \frac{\ln\tau}{\tau - 1}\right)\right]\left[1 - \frac{d\ln(w/A)}{d\ln\bar{Q}}\bigg)_{\Delta p}\right] - (1-n)\frac{d\ln(w/A)}{d\ln\bar{Q}}\bigg)_{\Delta p}\right\} d\tau}{\frac{h_1}{(h_1)_0}\left[\frac{\tau^{mn+1} - 1}{(\tau-1)(mn+1)}\right][\bar{Q}_0 - \bar{Q}]} \quad (25)$$

where $\mathfrak{J}$ is a dimensionless time defined

$$\mathfrak{J} \equiv \frac{1}{\mathfrak{m}}\,\frac{A}{\mathfrak{A} r_h}\left(\frac{h_1}{c}\right)_0 t \quad (26)$$

The result is shown in Fig. 5. The excursion is seen to start off rather slowly. If the total excursion takes 110 time units, then nothing much seems to be happening until during the last 50 units, when the excursion proceeds rapidly to the stable high flow-rate equilibrium point. The maximum rates of change seem to correspond to the occurrence of $\bar{Q}_{\max}$. For a graphite core with $\mathfrak{A}/A = 5$ operating at the stated conditions, each time unit is about 195 sec and the total excursion takes about 6 hr. Again this time is proportional to the heat capacity of the core and inverse to the heat-transfer rate.

The excursion to lower flow rate may also be calculated from Eq. (25). The results are shown in Fig. 6. This excursion proceeds very slowly. The time unit has the same length as in the example of Fig. 5. During 110 time units, the time for a complete excursion in Fig. 5, the excursion to lower flow rate has progressed only to a 50% increase in temperature ratio, a 30% decrease in Q, and a 60% decrease in flow rate.

The presented calculations are with respect to one operating point, that is a certain level of heat rate to the gas $\bar{Q}$, and an initial point that is quite remote from the neutral point. The characteristic times, generally speaking, are inverse to the level of $\bar{Q}$. Also, the number of time units for a complete excursion are considerably reduced when considering excursions from unstable equilibrium points that are closer to the neutral point. Nevertheless, these characteristic times are still of the order of seconds to minutes rather than fractions of a second.

More General Excursions

Excursions in general need not be at constant pressure drop. In fact, the data of Ref. 6 on the negatively sloped portion of the characteristic curve were obtained by fixing the flow rate (constant w/A) and increasing the electrical power input. The equivalent data of Refs. 7 and 8 were obtained in the inverse manner (fixing the electrical input and varying the flow rate). In both experiments the equilibrium points were approached apparently stably and the data recorded. One wonders therefore if the stability of an equilibrium point might not also depend on the direction of the excursion in the $\Delta p, (w/A)$ plane.

The transient equation that is most convenient for treating more general excursions is the following alternate form of Eq. (12):

$$\frac{d\bar{Q}}{dt}\left\{1 + (\tau - 1)\left[\frac{\bar{\bar{h}}T_1}{\bar{Q}}\frac{d(\bar{T}/T_1)}{d\tau} - \frac{\partial\,\widetilde{\ln h}}{\partial\tau}\right] \times \left[1 - \left(\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\text{act}}\right] - \frac{\partial\,\widetilde{\ln h}}{\partial\ln(w/A)}\left(\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\text{act}}\right\} = \frac{A}{m\mathfrak{A} r_h}\left(\frac{\bar{h}}{c}\right)\left[\frac{\bar{\varphi}\mathfrak{A} r_h}{A} - \bar{Q}\right] \quad (27)$$

where $[d\ln(w/A)/d\ln\bar{Q}]_{\text{act}}$ is the value of the indicated derivative along the actual excursion path. For example, at constant w/A, $[d\ln(w/A)/d\ln\bar{Q}]_{\text{act}} = 0$ whereas at constant $\bar{Q}$, the derivative becomes infinite.

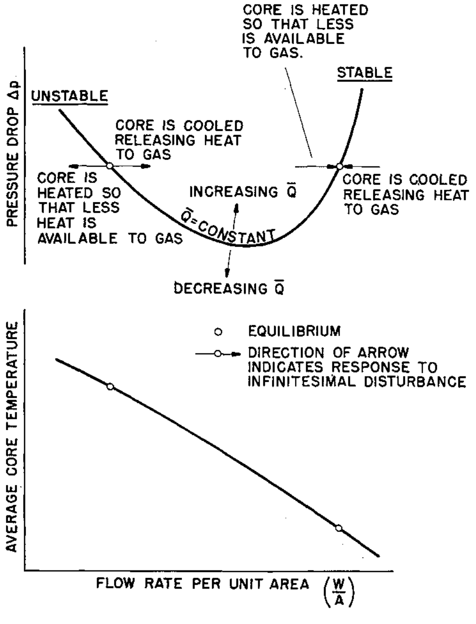

Fig. 2 Physical interpretation of stability criterion.

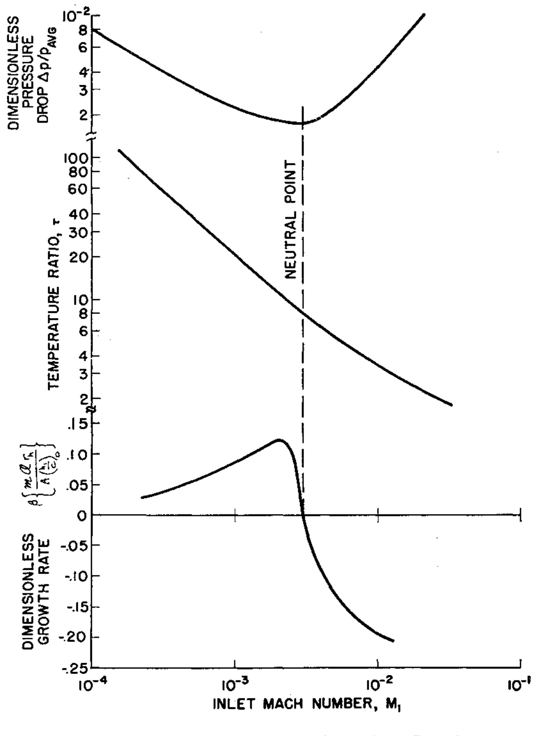

Fig. 3 Growth rates for infinitestimal disturbances. Steady-state conditions are those of Fig. 1 for $\bar{Q}$ = 0.2.

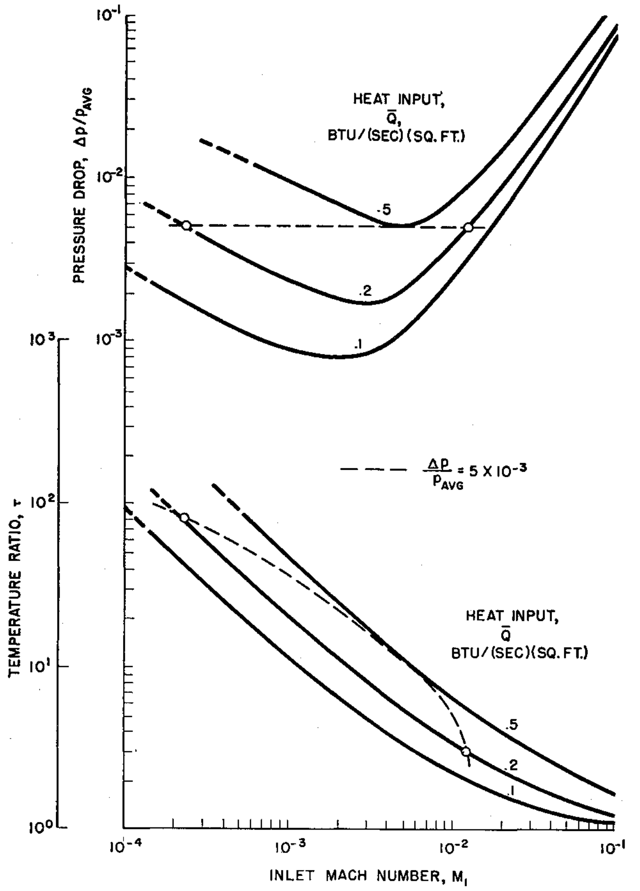

Fig. 4 Excursion paths for $\Delta p/p_{\text{avg}} = 5 \times 10^{-3}$. Steady-state conditions are those of Fig. 1.

By steps identical to those for constant Δp, the growth rate in the present case, β', is given by

$$\beta' = -(\gamma_0/\alpha_0') \tag{28}$$

where

$$\alpha_0' = \left\{1 + (\tau - 1)\left[\frac{\bar{\bar{h}}T_1}{\bar{Q}}\frac{d(\bar{T}/T_1)}{d\tau} - \frac{\partial\, \widetilde{\ln h}}{\partial\tau}\right] \times \left[1 - \left(\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\text{act}}\right] - \frac{\partial \ln h}{\partial \ln(w/A)}\left(\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\text{act}}\right\}_0 \tag{29}$$

The passage is thus unstable to excursions for which $\alpha_0' <$ 0, namely where [upon incorporating relation (B14)]

$$\left(\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\text{act}} > \frac{1}{1 - \dfrac{n}{1 + (\tau - 1)\{[\bar{\bar{h}}T_1/\bar{Q}][d(\bar{T}/T_1)/d\tau] - [\partial\, \widetilde{\ln h}/\partial\tau]\}}} \tag{30}$$

Since regardless of whether the flow is laminar or turbulent, the second term in the denominator of Eq. (30) is positive and small compared to one, the condition for instability may be abbreviated as

$$\left(\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\text{act}} > 1^+ \tag{31}$$

The application of this criterion to a reactor passage is shown qualitatively in Fig. 7. About each of three equilibrium points are shown the directions of $[d\ln(w/A)/d\ln\bar{Q}] =$ 0, 1^+, ∞. The stable directions are shaded. It is readily seen that all points on the curve, whether they be on the positively or negatively sloped branches, are stable to excursions at constant (w/A). Further, the points exhibit their now expected behavior to excursions at constant Δp. Lastly, each point exhibits a range of directions that are stable and a range of unstable directions. Even the positively sloped portion of the curve shows a narrow range of excursion directions to which the passage is unstable. The pertinent physical explanations are the same as given for Fig. 2 since the condition $[d\ln(w/A)/d\ln\bar{Q}] = 1^+$ is about that for which the average core temperature remains unchanged.

Also, extended excursion histories can be calculated in the present case for any assumed path in the $\Delta p,(w/A)$ plane.

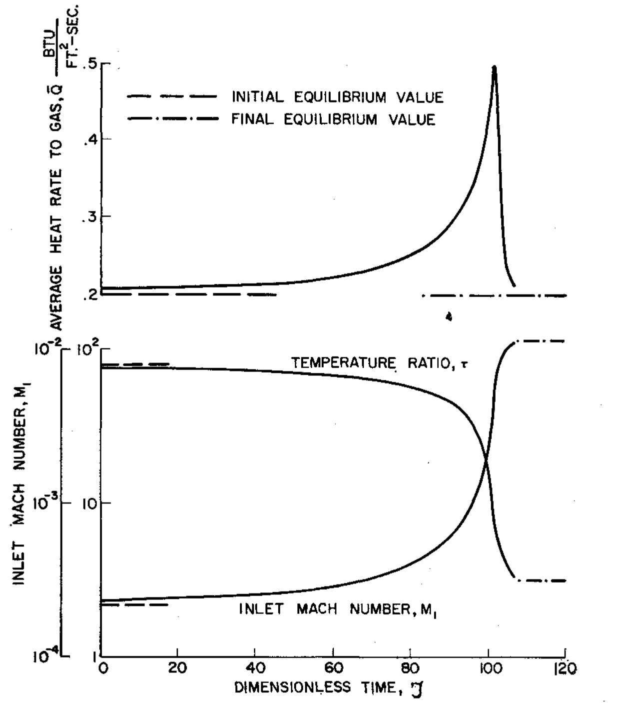

Fig 5 Time history of excursion to higher flow rate.

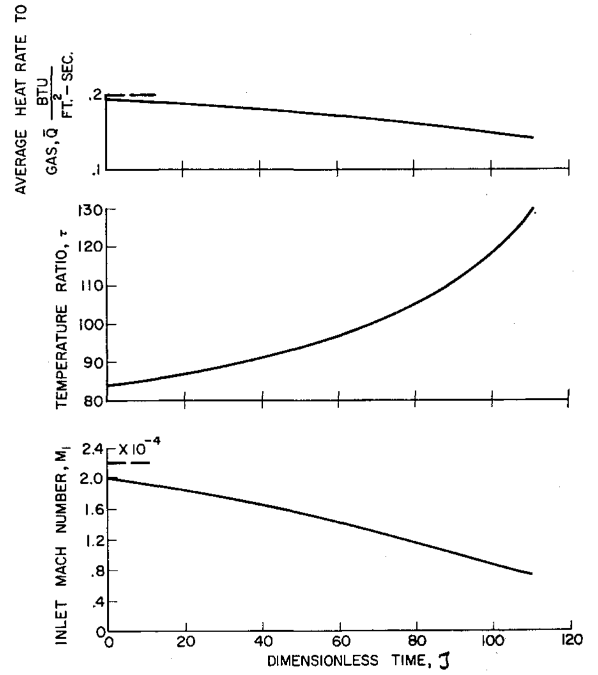

Fig. 6 Time history of excursion to lower flow rate.

The calculation procedure is analogous to that used earlier for excursions at constant pressure drop.

Concluding Remarks

In the present paper a time-dependent analysis has been presented for the laminar instability problem in heated constant-area reactor passages. It has been verified that at constant pressure drop, amplification of infinitesimal disturbances is obtained for $[\partial(\Delta p)/\partial(w/A)]_{\bar{Q}} < 0$. The growth rates of amplified infinitesimal disturbances at constant pressure drop are not monotonic but are maximum at a flow rate just below that for a neutral disturbance. All portions of the curve are stable to excursions at constant-weight flow. It is shown that the instability is not really a flow instability but rather is associated with the compatibility between the heat-transfer characteristics of the flow passage and the thermal characteristic of the reactor core.

There is nothing violent about this instability. The characteristic times are measured in seconds to minutes rather than fractions of a second. There is no oscillation but rather a steady procession away from unstable equilibrium points. It seems that because of this mildness, rather than avoid the region of instability one might try to conceive of operating in the "unstable" region achieving stabilization with external feedback controls.

It is hoped that some of the mystery associated with laminar instability has been removed and that this will lead to more straightforward thinking in this area of reactor design.

Appendix A: Evaluation of Integrals

The integrals in Eq. (11) that have to be evaluated are

$$① = \frac{1}{mL}\int_0^L \frac{h}{c}\,\varphi dx \tag{A1}$$

$$② = \frac{A}{m\alpha r_h L}\int_0^L \frac{h}{c}\,Q dx \tag{A2}$$

$$③ = \frac{1}{L}\int_0^L h\,\frac{\partial T}{\partial t}\,dx \tag{A3}$$

$$④ = \frac{1}{L}\int_0^L Q\,\frac{\partial \ln h}{\partial t}\,dx \tag{A4}$$

In performing the integrations it is convenient to describe Q as

$$Q = \bar{Q}q(x) \tag{A5}$$

where $q(x)$ is a normalized distribution:

$$\frac{1}{L}\int_0^L q(x)dx = 1$$

This is consistent with Eq. (7). It will be assumed at this point that the heat flux to the gas Q has the same axial distribution as the reactor heat release. Thus

$$\varphi = \bar{\varphi}\,q(x) \tag{A6}$$

where $\bar{\varphi}$ is the average value of φ defined as

$$\bar{\varphi} \equiv \frac{1}{L}\int_0^L \varphi dx \tag{A6a}$$

The functions $q(x)$ for some typical situations are
Uniform heating:

$$q(x) = 1 \tag{A7}$$

"Cosine" heating:

$$q(x) = \pi/2\,\sin(\pi x/L) \tag{A8}$$

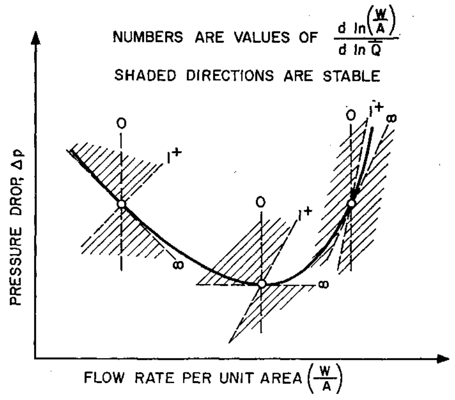

Fig. 7 Stability criterion for general excursions.

"Parabolic" heating:

$$q(x) = 6[(x/L) - (x/L)^2] \tag{A9}$$

With these definitions, the first two integrals become

$$① = (1/m)\bar{\varphi}(\bar{h}/c) \tag{A10}$$

$$② = (A/m\alpha r_h)\bar{Q}(\bar{h}/c) \tag{A11}$$

where

$$\left(\frac{\bar{h}}{c}\right) = \frac{1}{L}\int_0^L \left(\frac{h}{c}\right) q(x)dx \tag{A12}$$

Evaluation of the third integral requires an expression for the heat-transfer coefficient. From Colburn's form of Reynolds analogy

$$h = (w/A)c_p(f/2)Pr^{-2/3} \tag{A13}$$

where the friction factor f is given by the expression

$$f = \frac{C}{(\mathrm{Re})^n} = \frac{C\mu^n}{[4r_h(w/A)]^n} \tag{A14}$$

where C is a constant.
Thus

$$\begin{aligned} h &= \left(\frac{w}{A}\right) c_p \frac{C\mu_1^n}{2[4r_h(w/A)]^n}\left(\frac{\mu}{\mu_1}\right)^n Pr^{-2/3} \\ &= h_1(\mu/\mu_1)^n \end{aligned} \tag{A15}$$

and h_1 is the heat-transfer coefficient evaluated at the entrance temperature:

$$h_1 = \left(\frac{w}{A}\right)^{1-n} \frac{c_p C\mu_1^n}{2(4r_h)^n} Pr^{-2/3} \tag{A16}¶$$

The viscosity, in turn, is assumed to vary as temperature to a power ($\mu \sim T^m$) so that (A15) may be written

$$h(x,t) = h_1[T(x,t)/T_1]^{mn} \tag{A17}$$

¶ Reynolds' analogy properly yields a constant Nusselt number for fully developed laminar flow in constant-area passages. However, the value that results may be quantitatively off. For example, if $f = 16/Re$, the Nusselt number from (A16) is $8\ Pr^{1/3}$ whereas Eckert and Drake[9] indicate 4.36 for the Nusselt number for uniform heat addition. In performing excursion calculations, the heat-transfer coefficient h_1 may be evaluated using the best available information, and in fact for the laminar portions of the examples in the text $h_1 = 4.36\ k/D$ where D is the diameter of the circular flow passage. For turbulent flow, numerical adjustment is unnecessary.

Integral ③ can now be written

$$③ \equiv \frac{1}{L}\int_0^L h(x,t)\,\frac{\partial T(x,t)}{\partial t}\,dx = \frac{T_1h_1}{L}\int_0^L \left(\frac{\bar{T}}{T_1}\right)^{mn} \times \frac{\partial}{\partial t}\left(\frac{T}{T_1}\right) dx = \frac{T_1h_1}{L(mn+1)}\,\frac{d}{dt}\int_0^L \left(\frac{T}{T_1}\right)^{mn+1} dx = \frac{T_1h_1}{mn+1}\,\frac{d}{dt}\left[\left(\frac{\bar{T}}{T_1}\right)^{mn+1}\right] = T_1\bar{\bar{h}}_1\,\frac{d}{dt}\left(\frac{\bar{T}}{T_1}\right) \qquad \text{(A18)}$$

In Eq. (A18)

$$h \equiv h_1(\bar{T}/T_1)^{mn} \qquad \text{(A19)}$$

and

$$\left(\frac{\bar{T}}{\bar{T}_1}\right) \equiv \left[\frac{1}{L}\int_0^L \left(\frac{T}{T_1}\right)^{mn+1} dx\right]^{1/(mn+1)} \qquad \text{(A20)}$$

The ratio $(\bar{T}/T_1)$ is a function only of the inlet and outlet temperatures.

Assuming the inlet temperature to be fixed yields

$$③ = T_1\bar{\bar{h}}\,\frac{d}{dT_L}\left(\frac{\bar{T}}{T_1}\right)\frac{dT_L}{dt} \qquad \text{(A21)}$$

Since from Eq. (6)

$$\frac{dT_L}{dt} = \frac{L}{(w/A)c_pr_h}\,\frac{d\bar{Q}}{dt} - (T_L - T_1)\,\frac{d\ln(w/A)}{dt} \qquad \text{(A22)}$$

the resulting expression for integral ③ is

$$③ = T_1\bar{\bar{h}}\,\frac{d}{dT_L}\left(\frac{\bar{T}}{T_1}\right)\left\{\frac{L}{(w/A)c_pr_h}\,\frac{d\bar{Q}}{dt} - (T_L - T_1)\,\frac{d\ln(w/A)}{dt}\right\} \qquad \text{(A23)}$$

Integral ④ is evaluated as follows:

$$④ \equiv \frac{1}{L}\int_0^L \bar{Q}\,\frac{\partial \ln h}{\partial t}\,dx = \frac{1}{L}\int_0^L \bar{Q}\,\frac{\partial}{\partial t}\,[q(x)\ln h]dx$$

since $q(x)$ is not a function of time. This can be written simply as

$$④ = \bar{Q}(d/dt)(\widetilde{\ln h}) \qquad \text{(A24)}$$

where

$$\widetilde{\ln h} = \frac{1}{L}\int_0^L q(x)\ln h\,dx \qquad \text{(A25)}$$

But for fixed inlet temperature $\widetilde{\ln h} = \widetilde{\ln h}(w/A, T_L)$ so that with the help of Eq. (A22), Eq. (A24) may be written

$$④ = \bar{Q}\left[\left(\frac{\partial\,\widetilde{\ln h}}{\partial(w/A)} - \frac{(T_L - T_1)}{(w/A)}\,\frac{\partial\,\widetilde{\ln h}}{\partial T_L}\right)\frac{d(w/A)}{dt} + \frac{\partial\,\widetilde{\ln h}}{\partial T_L}\,\frac{L}{(w/A)c_pr_h}\,\frac{d\bar{Q}}{dt}\right] \qquad \text{(A26)}$$

The desired combination of integrals from Eq. (11) is

$$d\bar{Q}/dt = ① - ② - ③ + ④$$

which is

$$\frac{d\bar{Q}}{dt} = \frac{1}{\mathfrak{m}}\left(\frac{\bar{h}}{c}\right)\left[\bar{\varphi} - \frac{\bar{Q}A}{\mathfrak{A}r_h}\right] - T_1\bar{\bar{h}}\,\frac{d}{dT_L}\left(\frac{\bar{T}}{T_1}\right) \times \left\{\frac{L}{(w/A)c_pr_h}\,\frac{d\bar{Q}}{dt} - (T_L - T_1)\,\frac{d\ln(w/A)}{dt}\right\} + \bar{Q}\left[\left(\frac{\partial\,\widetilde{\ln h}}{\partial(w/A)} - \frac{T_L - T_1}{(w/A)}\,\frac{\partial\,\widetilde{\ln h}}{\partial T_L}\right)\frac{d(w/A)}{dt} + \frac{\partial\,\widetilde{\ln h}}{\partial T_L}\,\frac{L}{(w/A)c_pr_h}\,\frac{d\bar{Q}}{dt}\right] \qquad \text{(A27)}$$

With the aid of the following form of Eq. (4),

$$\frac{d(\Delta p)}{dt} = \left(\frac{\partial(\Delta p)}{\partial(w/A)}\right)_{\bar{Q}}\frac{d(w/A)}{dt} + \left(\frac{\partial(\Delta p)}{\partial\bar{Q}}\right)_{w/A}\frac{d\bar{Q}}{dt} \qquad \text{(A28)}$$

Eq. (A27) becomes

$$\frac{d\bar{Q}}{dt}\left\{1 + \left(\bar{\bar{h}}T_1\,\frac{d\left(\frac{\bar{T}}{T_1}\right)}{dT_L} - \bar{Q}\,\frac{\partial\,\widetilde{\ln h}}{\partial T_L}\right) \times \left[\frac{L}{\left(\frac{w}{A}\right)c_pr_h} + \frac{T_L - T_1}{\left(\frac{w}{A}\right)}\,\frac{\left(\frac{\partial(\Delta p)}{\partial\bar{Q}}\right)_{w/A}}{\left(\frac{\partial(\Delta p)}{\partial(w/A)}\right)_{\bar{Q}}}\right] + \bar{Q}\,\frac{\partial\,\widetilde{\ln h}}{\partial\left(\frac{w}{A}\right)}\,\frac{\left(\frac{\partial(\Delta p)}{\partial\bar{Q}}\right)_{w/A}}{\left(\frac{\partial(\Delta p)}{\partial(w/A)}\right)_{\bar{Q}}}\right\} = \frac{1}{\mathfrak{m}}\left(\frac{\bar{h}}{c}\right)\left[\bar{\varphi} - \frac{\bar{Q}A}{\mathfrak{A}r_h}\right] + \left\{\bar{\bar{h}}T_1\,\frac{d}{dT_L}\left(\frac{\bar{T}}{T_1}\right)\left[\frac{T_L - T_1}{\left(\frac{w}{A}\right)}\right] + \frac{\bar{Q}}{\left(\frac{w}{A}\right)} \times \left(\frac{\partial\,\widetilde{\ln h}}{\partial\left(\frac{w}{A}\right)} - (T_L - T_1)\,\frac{\partial\,\widetilde{\ln h}}{\partial T_L}\right)\right\}\frac{\frac{d(\Delta p)}{dt}}{\left(\frac{\partial(\Delta p)}{\partial(w/A)}\right)_{\bar{Q}}} \qquad \text{(A29)}$$

Appendix B: Transient Equation and Stability Criterion for Uniformly Heated Passages

The evaluation of the averaged quantities $\bar{\bar{h}}$, $(\bar{T}/T_1)$, $\widetilde{\ln h}$, and $(\bar{h}/c)$ that appear in Eqs. (12) and (14) requires first a description of the temperature distribution in the tube. From Eqs. (5a) and (A5)

$$\frac{\partial T}{\partial x} = \frac{\bar{Q}q(x)}{(w/A)c_pr_h} \qquad \text{(B1)}$$

and since for uniform heating $q(x) = 1$, then the temperature increases linearly with distance down the tube:

$$T = T_1\left(1 + \frac{T_L - T_1}{T_1}\,\frac{x}{L}\right) \qquad \frac{T}{T_1} = 1 + (\tau - 1)\xi \qquad \text{(B2)}$$

where $\xi = x/L$. From the definition (A20)

$$\left(\frac{\bar{T}}{T_1}\right) = \left[\int_0^1 [1 + (\tau - 1)\xi]^{mn+1}\,d\xi\right]^{1/(mn+1)} = \left[\frac{\tau^{mn+2} - 1}{(mn+2)(\tau - 1)}\right]^{1/(mn+1)} \qquad \text{(B3)}$$

and from (A19)

$$\bar{\bar{h}} = h_1\left[\frac{\tau^{mn+2} - 1}{(mn+2)(\tau - 1)}\right]^{mn/(mn+1)} \qquad \text{(B4)}$$

where h_1 is given by Eq. (A16).

The evaluation of $\widetilde{\ln h}$ is as follows: From Eq. (A15)

$$h = h_1(T/T_1)^{mn} \qquad \text{(B5)}$$

so that

$$\ln h = \ln h_1 + mn\ln(T/T_1)$$

and

$$\widetilde{\ln h} = \frac{1}{L}\int_0^L q(x)\ln h\,dx = \frac{1}{L}\int_0^L \ln h\,dx$$

$$= \ln h_1 + mn\int_0^1 \ln[1 + (\tau - 1)\xi]d\xi \qquad \text{(B6)}$$

$$= \ln h_1 + mn\left\{\frac{\tau\ln\tau}{\tau - 1} - 1\right\}$$

The correct determination of $(\bar{h}/c)$ requires a wall-temperature distribution so that c can be properly evaluated. From Eq. (9)

$$T_w = T + [\bar{Q}q(x)/h] \qquad \text{(B7)}$$

and for uniform heating

$$T_w = T_1[1 + (\tau - 1)\xi] + \frac{\bar{Q}}{h_1[1 + (\tau - 1)\xi]^{mn}} \qquad \text{(B8)}$$

The specific heat, in turn, is represented by an approximation of the Debye law,

$$c(T_w) = c_{\text{ref}}[1 - \tfrac{1}{20}(\theta_D/T_w)^2 + \ldots] \qquad \text{(B9)}$$

where c_{ref} is the specific heat at temperatures well above the Debye temperature θ_D. For the sake of simplicity, the specific heat of the reactor core, c, will be taken as constant in what follows. Then from (A12)

$$\left(\frac{\bar{h}}{c}\right) = \frac{h_1}{c}\int_0^1 [1 + (\tau - 1)\xi]^{mn}\,d\xi$$

$$= \frac{h_1}{c}\left[\frac{\tau^{mn+1} - 1}{(\tau - 1)(mn + 1)}\right] \qquad \text{(B10)}$$

Also required are

$$\frac{d(\bar{T}/T_1)}{d\tau} = \left[\frac{\tau^{mn+2} - 1}{(mn + 2)(\tau - 1)}\right]^{-mn/mn+1} \times$$

$$\left[\frac{(\tau - 1)(mn + 2)\tau^{mn+1} - (\tau^{mn+2} - 1)}{(mn + 1)(mn + 2)(\tau - 1)^2}\right] \qquad \text{(B11)}$$

$$\bar{\bar{h}}\frac{d(\bar{T}/T_1)}{d\tau} = h_1\left[\frac{(mn + 2)(\tau - 1)\tau^{mn+1} - (\tau^{mn+2} - 1)}{(mn + 1)(mn + 2)(\tau - 1)^2}\right] \qquad \text{(B12)}$$

$$\frac{\partial\,\widetilde{\ln h}}{\partial\tau} = mn\left[\frac{(\tau - 1) - \ln\tau}{(\tau - 1)^2}\right] \qquad \text{(B13)}$$

$$\frac{\partial\ln h}{\partial\ln(w/A)} = \frac{\partial\ln h_1}{\partial\ln(w/A)} = 1 - n \qquad \text{(B14)}$$

These have been obtained by performing the indicated differentiations.

Substitution of (B11–B14) into Eq. (14) yields the equation governing an excursion from equilibrium;

$$\frac{d\bar{Q}}{dt}\left\{1 + \left[\frac{h_1T_1}{\bar{Q}}\left(\frac{(mn + 2)(\tau - 1)\tau^{mn+1} - (\tau^{mn+2} - 1)}{(mn + 1)(mn + 2)(\tau - 1)}\right) - mn\left(1 - \frac{\ln\tau}{\tau - 1}\right)\right]\left[1 - \left.\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\Delta p}\right] - (1 - n)\left.\frac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\Delta p}\right\} = \frac{1}{\mathfrak{m}}\frac{A}{\mathfrak{A}r_h}\left(\frac{h_1}{c}\right) \times \left[\frac{\tau^{mn+1} - 1}{(\tau - 1)(mn + 1)}\right]\left[\frac{\bar{\varphi}\mathfrak{A}r_h}{A} - \bar{Q}\right] \qquad \text{(B15)}$$

The growth rate β of an infinitesimal disturbance is, from Eq. (19),

$$\beta = \frac{-\dfrac{1}{\mathfrak{m}}\dfrac{A}{\mathfrak{A}r_h}\left(\dfrac{h_1}{c}\right)_0\left[\dfrac{\tau^{mn+1} - 1}{(\tau - 1)(mn + 1)}\right]_0}{\left\{1 + \left[\dfrac{h_1T_1}{\bar{Q}}\left(\dfrac{(mn + 2)(\tau - 1)\tau^{mn+1} - (\tau^{mn+2} - 1)}{(mn + 1)(mn + 2)(\tau - 1)}\right) - mn\left(1 - \dfrac{\ln\tau}{\tau - 1}\right)\right]\left[1 - \left.\dfrac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\Delta p}\right] - (1 - n)\left.\dfrac{d\ln(w/A)}{d\ln\bar{Q}}\right)_{\Delta p}\right\}} \qquad \text{(B16)}$$

where the subscript zero indicates that the quantities are to be evaluated at the equilibrium point being studied.

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